CAT(0) SPACES ON WHICH A CERTAIN TYPE OF SINGULARITY IS BOUNDED

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Abstract. In this paper, we will consider a family $\mathcal{Y}$ of complete CAT(0) spaces such that the tangent cone TC_pY at each point $p \in Y$ of each $Y \in \mathcal{Y}$ is isometric to a (finite or infinite) product of the Euclidean cones $\mathrm{Cone}(X_\alpha)$ over elements X_α of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of CAT(1) spaces. Each element of such $\mathcal{Y}$ is a space presented by Gromov [4] as an example of a "CAT(0) space with "bounded" singularities". We will show that the Izeki-Nayatani invariants of spaces in such a family are uniformly bounded from above by a constant strictly less than 1.

1. Introduction

In [4], Gromov introduced the term "CAT(0) space with 'bounded' singularities", and remarked that there exist infinite groups which admit no uniform embeddings into such a space. He used this terminology without providing its precise definition, but as examples of such spaces, he presented CAT(0) spaces Y such that the tangent cone TC_pY at each point $p \in Y$ is isometric to a (finite or infinite) product of Euclidean cones $\mathrm{Cone}(X_\alpha)$ over elements X_α of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of CAT(1) spaces.

On the other hand, Izeki and Nayatani [5] defined an invariant $\delta(Y) \in [0,1]$ of a complete CAT(0) space Y. And some general results for CAT(0) spaces whose Izeki-Nayatani invariants are bounded from above were proved by Izeki, Kondo, and Nayatani ([5], [6], [7], [8], [9]). Group Γ is said to have the *fixed-point property* for a metric space Y, if for any group homomorphism $\rho : \Gamma \to \mathrm{Isom}(Y)$ there exists a point $p \in Y$ such that $\rho(\gamma)p = p$ for all $\gamma \in \Gamma$. Izeki, Kondo and Nayatani [7] proved that a random group of Gromov's graph model has the fixed-point property for all elements Y of a family $\mathcal{Y}$ of CAT(0) spaces whose Izeki-Nayatani invariants are uniformly bounded from above by a constant strictly less than 1:

$$\sup\{\delta(Y) \mid Y \in \mathcal{Y}\} < 1.$$

Moreover, it is straightforward to see that an expander admits no uniform embedding into a complete CAT(0) space Y with $\delta(Y) < 1$ (see [9]). Combining this with Gromov's argument in [4], the existence of infinite groups which admit no uniform embeddings into a space Y with $\delta(Y) < 1$ follows. This seems to suggest that the

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Izeki-Nayatani invariant measures a certain type of "singularity" similar to Gromov's notion.

Although these general results were proved, the computation of the Izeki-Nayatani invariant is difficult. It is still unclear what kind of CAT(0) spaces Y or families $\mathcal{Y}$ of CAT(0) spaces have the boundedness property as above. It had been even unknown whether there exists a complete CAT(0) space Y with $\delta(Y) = 1$ or not, until Kondo [9] showed the existence of CAT(0) spaces with $\delta = 1$ fairly recently.

In this paper, we prove the following theorem.

Theorem 1.1. *Let $\mathcal{Y}$ be a family of complete* CAT(0) *spaces such that the tangent cone TC_pY at each point $p \in Y$ on each $Y \in \mathcal{Y}$ is isometric to a (finite or infinite) product of the Euclidean cones* $\mathrm{Cone}(X_\alpha)$ *over elements X_α of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of complete* CAT(1) *spaces. Then we have*

$$\sup_{Y \in \mathcal{Y}} \delta(Y) < 1.$$

Here, we use the word *product* of Euclidean cones $T_1, T_2, \dots$ in the sense of ℓ^2-product of the pointed metric spaces $(T_1, O_1), (T_2, O_2), \dots$, where each O_n is the cone point of T_n. That is, the product T of the cones $T_1, T_2, \dots$ consists of all sequences $(x_n)_n$ such that $x_n \in T_n$ and $\sum_n d_n(O_n, x_n)^2 < \infty$, and T is equipped with the metric function d defined by

$$d(x,y)^2 = \sum_{n=1}^{\infty} d_n(x_n, y_n)^2$$

for any $x = (x_1, x_2, \dots) \in T$ and any $y = (y_1, y_2, \dots) \in T$, where d_n is the metric function on T_n for each n. Then, T also has a cone structure with the cone point $O = (O_1, O_2, \dots)$. And completeness and CAT(0) condition are preserved by this construction.

Combining Theorem 1.1 with the general results mentioned above, we have the following corollary.

Corollary 1.2. **(i)** *If Y is a complete* CAT(0) *space such that the tangent cone at each point $y \in Y$ is isometric to a (finite or infinite) product of Euclidean cones* $\mathrm{Cone}(X_\alpha)$ *over elements X_α of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of* CAT(1) *spaces, then there exists infinite groups which admit no uniform embeddings into Y.* **(ii)** *There exist infinite groups which has the fixed-point property for all elements Y in such a family $\mathcal{Y}$ as in Theorem 1.1.*

Here, **(i)** has already been remarked in [4]. And **(ii)** follows from the general result in [7]. **(ii)** can be stated in terms of random groups(see [7]).

In the end of this paper, we claim that by the same technique used in the proof of Theorem 1.1, we can prove a more general statement, which includes Theorem 1.1 as a special case (Proposition 5.4).

2. Preliminaries on CAT(0) spaces

In this section we recall some basic definitions and facts concerning CAT(0) spaces. For a detailed exposition, we refer the reader to [1], [2] or [11].

For $\kappa > 0$ let M^2_κ denote the simply connected, complete 2-dimensional Riemannian manifold of constant Gaussian curvature κ, and let d_κ be its distance function. Let $D_\kappa \in (0, \infty]$ be the diameter of M^2_κ.

Let (Y, d_Y) be a metric space. A *geodesic* in Y is an isometric embedding γ of a closed interval $[a, b]$ into Y. A *geodesic triangle* in Y is a triple $\triangle = (\gamma_1, \gamma_2, \gamma_3)$ of geodesics $\gamma_i : [a_i, b_i] \to Y$ such that

$$\gamma_1(b_1) = \gamma_2(a_2), \quad \gamma_2(b_2) = \gamma_3(a_3), \quad \gamma_3(b_3) = \gamma_1(a_1).$$

If $\triangle$ has a perimeter less than $2D_\kappa$: $\sum_{i=1}^{3} |b_i - a_i| < 2D_\kappa$, then there is a geodesic triangle

$$\triangle^\kappa = (\gamma_1^\kappa, \gamma_2^\kappa, \gamma_3^\kappa), \quad \gamma_i : [a_i, b_i] \to M^2_\kappa$$

in M^2_κ , which has the same side lengths as $\triangle$. This triangle $\triangle^\kappa$ is unique up to isometry of M^2_κ, and we call it the *comparison triangle* of $\triangle$ in M^2_κ. Then $\triangle$ is said to be κ-*thin* if

$$d_Y(\gamma_i(s), \gamma_j(t)) \le d_\kappa(\gamma_i^\kappa(s), \gamma_j^\kappa(t))$$

whenever $i, j \in \{1, 2, 3\}$ and $s \in [a_i, b_i]$, and $t \in [a_j, b_j]$.

Definition 2.1. A metric space (Y, d) is called a *CAT*(κ) *space*, if for any pair of points $p, q \in Y$ with $d(p, q) < D_\kappa$ there exists a geodesic from p to q, and any geodesic triangle in Y with perimeter$< 2D_\kappa$ is κ-thin.

Next, we recall the definition of the Euclidean cone. Let (X, d_X) be a metric space. The cone $\mathrm{Cone}(X)$ over X is the quotient of the product $X \times [0, \infty)$ obtained by identifying all points in $X \times \{0\} \subset X \times [0, \infty)$. The point represented by $(x, 0)$ is called the *cone point* of $\mathrm{Cone}(X)$ and we will denote this point by $O_{\mathrm{Cone}(X)}$ in this paper. The cone distance $d_{\mathrm{Cone}(X)}(v, w)$ between two points $v, w \in \mathrm{Cone}(X)$ represented by $(x, t), (y, s) \in X \times [0, \infty)$ respectively, is defined by

$$d_{\mathrm{Cone}(X)}(v, w) = \sqrt{t^2 + s^2 - 2ts \cos(\min\{\pi, d_X(x, y)\})}.$$

Then $(\mathrm{Cone}(X), d_{\mathrm{Cone}(X)})$ is a metric space, and we call it the *Euclidean cone* over (X, d_X). It is known that a metric space (X, d_X) is a CAT(1) space if and only if $(\mathrm{Cone}(X), d_{\mathrm{Cone}(X)})$ is a CAT(0) space.

Suppose that Y is a CAT(0) space. Then by the definition of CAT(0) space, there is a unique geodesic joining any pair of points in Y. So, for any triple of points (p, q, r) in Y, it makes sense to denote by $\triangle(p, q, r)$ the geodesic triangle consisting of three geodesics joining each pair of the three points.

Let $\gamma : [a, b] \to Y$, $\gamma' : [a', b'] \to Y$ be two geodesics in a CAT(0) space Y such that

$$\gamma(a) = \gamma'(a') = p \in Y.$$

We define the *angle* $\angle_p(\gamma, \gamma')$ between γ, γ' as

$$\angle_p(\gamma, \gamma') = \lim_{t \to a, t' \to a'} \angle_p^0(\gamma(t), \gamma(t')),$$

where $\angle_p^0(\gamma(t),\gamma(t'))$ is the corresponding angle of the comparison triangle of $\triangle(p,\gamma(t),\gamma'(t'))$ in $M_0^2=\mathbb{R}^2$. The existence of the limit follows from the definition of CAT(0) space.

Definition 2.2. Let (Y,d_Y) be a complete CAT(0) space, and let $p\in Y$. We denote by $(S_pY)^\circ$ the set of all geodesics $\gamma:[a,b]\to Y$ such that $\gamma(a)=p$. Then the angle $\angle_p$ defines a pseudometric on $(S_pY)^\circ$. The *space of directions* S_pY at p is the metric completion of the quotient space of (S_pY) where we identify any $x,y\in S_pY$ with $\angle_p(x,y)=0$. We define the *tangent cone* TC_pY of Y at p to be the Euclidean cone $\mathrm{Cone}(S_pY)$ over the space of directions at p.

If (Y,d_Y) is a complete CAT(0) space and if $p\in Y$, then it can be proved that the space of directions S_pY at p is a complete CAT(1) space. Hence, the tangent cone TC_pY at p is a complete CAT(0) space.

Finally, we recall some basic notions and facts about probability measures on a metric space (Y,d_Y). In this paper, we will treat only finitely supported measures. Measure ν on Y is *finitely supported* if there exists a finite subset $S\subset Y$ such that $\nu(Y\backslash S)=0$. We call the minimal subset S with such a property the *support* of ν, and denote it by $\mathrm{supp}(\nu)$. We denote by $\mathcal{P}(Y)$ the set of all finitely supported probability measures on Y. If $\mathrm{supp}(\nu)=\{p_1,\dots,p_n\}$, then ν can be represented as

$$\nu=\sum_{i=1}^n t_i\mathrm{Dirac}_{p_i} \tag{2.1}$$

by nonnegative real numbers $t_1,\dots,t_n$ with $\sum_{i=1}^n t_i=1$, where Dirac_{p_i} stands for the Dirac measure at $p_i\in Y$. We will also use the notation $\mathcal{P}'(Y)$ to denote the subset of $\mathcal{P}(Y)$ consisting of all measures whose supports contain at least two points. Let Z be a set and let $\phi:Y\to X$ be a map. Then for any $\nu\in\mathcal{P}(Y)$, we define the *pushforward* measure $\phi_*\mu$ on X as

$$\phi_*\nu(A)=\mu\left(\phi^{-1}(A)\right),\quad A\subset X$$

If we write ν as in the form (2.1), we can write $\phi_*\nu$ as

$$\phi_*\nu=\sum_{i=1}^n t_i\mathrm{Dirac}_{\phi(p_i)}$$

If (Y,d_Y) is a complete CAT(0) space, and if $\nu\in\mathcal{P}(Y)$, there exists a unique point $\mathrm{bar}(\nu)\in Y$ which minimizes the function

$$y\mapsto\int_Y d(y,z)^2\nu(dz)$$

defined on Y. This point is called the *barycenter* of ν. We refer the reader to [11] for the existence and uniqueness of barycenter.

3. Hilbert sphere valued maps and an invariant of a CAT(1) space

In this section, we define a certain invariant of complete CAT(1) spaces. First we set up some notations for Hilbert sphere valued maps on CAT(1) spaces. Let $\mathcal{H}$

be a real Hilbert space, and let $\phi : X \to \mathcal{H}$ be a map whose image is contained in the unit sphere in $\mathcal{H}$. Thus $\|\phi(x)\| = 1$ for all $x \in X$. Let $\mu \in \mathcal{P}(X)$ be a finitely supported probability measure on X. We define the vector $\mathbb{E}_\mu[\phi] \in \mathcal{H}$ as

$$\mathbb{E}_\mu[\phi] = \int_X \phi(x)\mu(dx).$$

And if the vector $\mathbb{E}_\mu[\phi]$ is not the zero vector, we denote by $\tilde{\mathbb{E}}_\mu[\phi]$ the unit vector parallel to $\mathbb{E}_\mu[\phi]$:

$$\tilde{\mathbb{E}}_\mu[\phi] = \frac{1}{\|\mathbb{E}_\mu[\phi]\|}\mathbb{E}_\mu[\phi].$$

Then the value $\|\mathbb{E}_\mu[\phi]\| \in [0,1]$ amounts to a sort of concentration of the pushforward measure $\phi_*\mu$ around $\tilde{\mathbb{E}}_\mu[\phi]$ on the unit sphere. By simple calculation, we have

$$\|\mathbb{E}_\mu[\phi]\| = \int_X \langle \tilde{\mathbb{E}}_\mu[\phi], \phi(x)\rangle \mu(dx) \tag{3.1}$$

whenever $\|\mathbb{E}_\mu[\phi]\| \neq 0$.

Now we define an invariant of a complete CAT(1) space by using the notations introduced above. This invariant is designed for estimating the Izeki-Nayatani invariant of a CAT(0) space, whose definition will be recalled in the next section.

Definition 3.1. Let (X, d_X) be a metric space, and let $\mu \in \mathcal{P}(X)$. We define $\tilde{\delta}(\mu) \in [0,1]$ to be

$$\tilde{\delta}(\mu) = \inf_\phi \|\mathbb{E}_\mu[\phi]\|^2,$$

where the infimum is taken over all maps $\phi : X \to \mathcal{H}$ to some Hilbert space $\mathcal{H}$ such that

$$\|\phi(x)\| = 1, \quad \angle(\phi(x), \phi(y)) \le d_X(x,y) \tag{3.2}$$

for any $x, y \in X$. Here and henceforth, we denote the angle between two vectors v, w in any Hilbert space by $\angle(v,w)$.

Suppose (X, d_X) is a complete CAT(1) space and $\iota : X \to \mathrm{Cone}(X)$ is the canonical inclusion of X into its Euclidean cone. Then, we define $\tilde{\delta}(X)$ to be

$$\tilde{\delta}(X) = \sup\{\tilde{\delta}(\mu) \mid \mu \in \mathcal{P}(X), \mathrm{bar}(\iota_*\mu) = O_{\mathrm{Cone}(X)}\}.$$

When there is no measure satisfying such a condition, we define $\tilde{\delta}(X) = -\infty$.

To estimate this invariant in the proceeding sections, we will use the following fact:

Lemma 3.2. *Let (X, d_X) be a complete* CAT(1) *space. For $v, w \in \mathrm{Cone}(X)$ represented by $(x,t), (y,s) \in X \times \mathbb{R}$ respectively, we set*

$$\langle v, w\rangle = ts\cos(\min\{\pi, d_X(x,y)\}).$$

Then for any $\nu \in \mathcal{P}(\mathrm{Cone}(X))$ the following two conditions are equivalent:

- **(i):** $\mathrm{bar}(\nu) = O_{\mathrm{Cone}(X)}$.
- **(ii):** $\int_{\mathrm{Cone}(X)} \langle E_x, v\rangle\, \nu(dv) \le 0$, *whenever $x \in X$ and E_x is an element of* $\mathrm{Cone}(X)$ *represented by $(x,1)$.*

Proof. For $w \in \mathrm{Cone}(X)$ represented by $w = (y, s) \in X \times \mathbb{R}$, we write $\|w\| = s$. Fix $x \in X$ and let v_t be an element of $\mathrm{Cone}(X)$ represented by $(x, t) \in X \times \mathbb{R}$. Suppose that $\mathrm{bar}(\nu) = O_{\mathrm{Cone}(X)}$. Then the function

$$
\begin{aligned}
(3.3) \qquad F_x(t) &= \int_{\mathrm{Cone}(X)} d_{\mathrm{Cone}(X)}(v_t, w)^2 \nu(dw) \\
&= \int_{\mathrm{Cone}(X)} \left\{ t^2 + \|w\|^2 - 2t\langle E_x, w\rangle \right\} \nu(dw),
\end{aligned}
$$

defined on $[0, \infty)$ must attain its minimum at $t = 0$. This happens if and only if

$$F_x'(t) = 2\left(t - \int_{\mathrm{Cone}(X)} \langle E_x, w\rangle \, \nu(dw) \right) \geq 0.$$

for all $t \in \mathbb{R}$. So **(ii)** follows.

Conversely, if **(ii)** holds, then the function F_x on $[0, \infty)$ as (3.3) attains its minimum at $t = 0$ for each $x \in X$. And it is easily seen that $\mathrm{bar}(\nu) = O_{\mathrm{Cone}(X)}$. $\square$

In the final section, we will use this lemma in the following form.

Corollary 3.3. *Let (X, d_X) be a complete* CAT(1) *space, and let $\iota : X \to \mathrm{Cone}(X)$ be the canonical inclusion. If $\mu \in \mathcal{P}(X)$ satisfies $\mathrm{bar}(\iota_*\mu) = O_{\mathrm{Cone}(X)}$, then we have*

$$\mu\left(\left\{ y \in X \;\middle|\; d_X(x, y) \leq \theta \right\} \right) \leq \frac{1}{1 + \cos\theta}$$

for any $x \in X$ and any $0 \leq \theta < \frac{\pi}{2}$. In particular, we have

$$\mu\left(\left\{ y \in X \;\middle|\; d_X(x, y) \leq \frac{\pi}{3} \right\} \right) \leq \frac{2}{3}$$

for all $x \in X$.

Proof. Suppose there is $x_0 \in X$ such that

$$\mu\left(\left\{ y \in X \;\middle|\; d_X(x_0, y) \leq \theta \right\} \right) > \frac{1}{1 + \cos\theta}.$$

Then we would have

$$
\begin{aligned}
\int_X \cos\left(\min\{\pi, d_X(x_0, x)\}\right) \mu(dx) &= \int_{\{x \in X \,|\, d_X(x, x_0) \leq \theta\}} \cos\left(\min\{\pi, d_X(x_0, x)\}\right) \mu(dx) \\
&\quad + \int_{X \setminus \{x \in X \,|\, d_X(x, x_0) \leq \theta\}} \cos\left(\min\{\pi, d_X(x_0, x)\}\right) \mu(dx) \\
&> \cos\theta \times \frac{1}{1 + \cos\theta} + (-1) \times \left(1 - \frac{1}{1 + \cos\theta} \right) \\
&= 0.
\end{aligned}
$$

This implies $\mathrm{bar}(\iota_*\mu) \neq O_{\mathrm{Cone}(X)}$ by Lemma 3.2, which is a contradiction. $\square$

4. Izeki-Nayatani invariant

In this section, we recall the definition of the invariant δ of a complete CAT(0) space introduced by Izeki and Nayatani [5]. We will then derive a relation between δ and the invariant $\tilde{\delta}$ of a complete CAT(1) space defined in the previous section. More information about the Izeki-Nayatani invariant δ can be found in [5], [6], [7], [8] and [10].

Definition 4.1 ([5])**.** Let (Y, d_Y) be a complete CAT(0) space. Recall that $\mathcal{P}'(Y)$ is the subset of $\mathcal{P}(Y)$ consisting of all measures whose supports contain at least two points. For any $\nu \in \mathcal{P}'(Y)$, we define $\delta(\nu)$ to be

$$\delta(\nu) = \inf_{\phi} \frac{\| \int_Y \phi(p)\nu(dp)\|^2}{\int_Y \|\phi(p)\|^2 \nu(dp)},$$

where the infimum is taken over all maps $\phi : \mathrm{supp}(\nu) \to \mathcal{H}$ from the support of ν to some Hilbert space $\mathcal{H}$ such that

$$\|\phi(p)\| = d(\mathrm{bar}(\nu), p), \tag{4.1}$$

$$\|\phi(p) - \phi(q)\| \leq d(p, q) \tag{4.2}$$

for all $p, q \in \mathrm{supp}(\nu)$. Then the Izeki-Nayatani invariant $\delta(Y)$ of Y is defined by

$$\delta(Y) = \sup \{\delta(\nu) \mid \nu \in \mathcal{P}'(Y)\}.$$

By definition, we have $0 \leq \delta(\nu) \leq 1$ and $0 \leq \delta(Y) \leq 1$. When Y is a Euclidean cone, we define $\delta(Y, O_Y) \in [0, 1]$ to be

$$\delta(Y, O_Y) = \sup \{\delta(\nu) \mid \nu \in \mathcal{P}'(Y), \mathrm{bar}(\nu) = O_Y\},$$

where O_Y is the cone point of Y. When there is no measure satisfying such a condition, we define $\delta(Y, O_Y) = -\infty$. The following lemma is shown in [5].

Lemma 4.2 ([5])**.** *Suppose that Y is a complete* CAT(0) *space, and $\nu \in \mathcal{P}'(Y)$. Then we have*

$$\delta(\nu) \leq \delta(TC_{\mathrm{bar}(\nu)}Y,\, O_{TC_{\mathrm{bar}(\nu)}Y}).$$

In particular, we have

$$\delta(Y) \leq \sup\{\delta(TC_pY,\, O_{TC_pY}) \mid p \in Y\}.$$

The following lemma is a slight generalization of Proposition 6.5 in [5].

Lemma 4.3. *Let $(T_1, d_1), (T_2, d_2), (T_3, d_3), \ldots$ be complete* CAT(0) *spaces which are isometric to Euclidean cones, and let $O_1, O_2, \ldots$ be their cone points respectively. Let T be the cone obtained as the product of $T_1, T_2, \ldots$ with the cone point $O = (O_1, O_2, \ldots)$. Then we have*

$$\delta(T, O) = \sup_n \delta(T_n,\, O_n).$$

Proof. The following proof is almost the same argument as in the proof of Proposition 6.5 in [5]. We however include it for the sake of completeness.

First, the inequality $\delta(T,O) \geq \sup_n \delta(T_n, O_n)$ is obvious. Because we have the canonical isometric embedding $\mathcal{I}_n : T_n \to T$ for each n, and for each $\mu \in \mathcal{P}'(T_n)$ with $\mathrm{bar}(\mu) = O_n$, it is easy to see that $\mathrm{bar}(\mathcal{I}_{n*}\mu) = O$ and $\delta(\mu) = \delta(\mathcal{I}_{n*}\mu)$.

Let

$$\mu = \sum_{i=1}^{m} t_i \mathrm{Dirac}_{v_i} \in \mathcal{P}'(T)$$

be an arbitrary measure in $\mathcal{P}'(T)$ with $\mathrm{bar}(\mu) = O$, where $v_1, \ldots, v_m \in T$ and $t_1, \ldots, t_m > 0$ with $\sum_{i=1}^m t_i = 1$. Write $v_i = (v_i^{(1)}, v_i^{(2)}, \ldots)$ and let

$$\mu_n = \sum_{i=1}^{m} t_i \mathrm{Dirac}_{v_i^{(n)}} \in \mathcal{P}'(T_n), \quad n = 1, 2, \ldots.$$

Then $\mathrm{bar}(\mu_n) = O_n$ for each n. Because if we have $\mathrm{bar}(\mu_n) \neq O_n$ for some n, it is easy to show that

$$\int_T d(w, B)^2 \mu(dw) < \int_T d(w, O)^2 \mu(dw),$$

where $B \in T$ is a point in T such that all of its components are the cone points but $\mathrm{bar}(\mu_n)$ for the n-th component, and it contradicts the assumption that $\mathrm{bar}(\mu) = O$.

Let $\varepsilon > 0$ be an arbitrary positive number. By the definition of $\delta(T_n, O_n)$, there exists a map $\phi_n : \mathrm{supp}(\mu_n) \to \mathcal{H}_n$ from the support of μ_n to some Hilbert space $\mathcal{H}_n$ with the properties (4.1) and (4.2) with respect to μ_n, satisfying

$$\frac{\| \int_{T_n} \phi_n(v) \mu_n(dv) \|^2}{\int_{T_n} \|\phi_n(v)\|^2 \mu_n(dv)} \leq \delta(T_n, O_n) + \varepsilon.$$

We define a map $\phi : \mathrm{supp}(\mu) \to \mathcal{H}$ from the support of μ to the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$ to be

$$\phi(v_i) = \left(\phi_1(v_i^{(1)}), \phi_2(v_i^{(2)}), \ldots \right), \quad i = 1, \ldots, m.$$

Then it is straightforward to see that ϕ satisfies the properties (4.1) and (4.2) with respect to μ. And we have

$$\begin{aligned} \delta(\mu) \leq \frac{\| \int_T \phi(v) \mu(dv) \|^2}{\int_T \|\phi(v)\|^2 \mu(dv)} &= \frac{\sum_{n=1}^{\infty} \| \sum_{i=1}^m t_i \phi_n(v_i^{(n)}) \|^2}{\sum_{n=1}^{\infty} \sum_{i=1}^m t_i \|\phi_n(v_i^{(n)})\|^2} \\ &\leq \sup_n \frac{\| \sum_{i=1}^m t_i \phi_n(v_i^{(n)}) \|^2}{\sum_{i=1}^m t_i \|\phi_n(v_i^{(n)})\|^2} \leq \sup_n \left(\delta(T_n, O_n) + \varepsilon \right). \end{aligned}$$

Since this holds for an arbitrary $\varepsilon > 0$ and an arbitrary $\mu \in \mathcal{P}'(T)$ with $\mathrm{bar}(\mu) = O$, we have $\delta(T, O) \leq \sup_n \delta(T_n, O_n)$. □

For a CAT(1) space X, we prove the following relation between $\delta(\mathrm{Cone}(X), O_{\mathrm{Cone}(X)})$ and $\tilde{\delta}(X)$

Proposition 4.4. *Let (X, d_X) be a complete* CAT(1) *space. Then we have*

$$\delta(\mathrm{Cone}(X), O_{\mathrm{Cone}(X)}) \leq \tilde{\delta}(X).$$

Before proving Proposition 4.4, we establish the following two lemmas.

Lemma 4.5. *Let (X, d_X) be a complete* CAT(1) *space. Let*

$$\nu = \sum_{i=1}^{m} t_i \mathrm{Dirac}_{v_i} \in \mathcal{P}'(\mathrm{Cone}(X)),$$

where $v_i \in \mathrm{Cone}(X)$ for $i = 1, \ldots, m$ and $t_1, \cdots, t_m > 0$ with $\sum_{i=1}^{m} t_i = 1$. Suppose that $\mathrm{bar}(\nu) = O_{\mathrm{Cone}(X)}$. If $v_1 = O_{\mathrm{Cone}(X)}$ and if

$$\nu' = \sum_{i=2}^{m} \frac{t_i}{1 - t_1} \mathrm{Dirac}_{v_i},$$

then $\mathrm{bar}(\nu') = O_{\mathrm{Cone}(X)}$ and $\delta(\nu) \leq \delta(\nu')$.

Proof. The former assertion follows immediately from Lemma 3.2. Let $\phi' : \mathrm{supp}(\nu') \to \mathcal{H}$ be a map from the support of ν' to some Hilbert space $\mathcal{H}$ satisfying (4.1) and (4.2) with respect to ν'. Define $\phi : \mathrm{supp}(\nu) \to \mathcal{H}$ by

$$\begin{aligned} &\phi(v_1) = 0, \\ &\phi(v_i) = \phi'(v_i), \quad i = 2, \cdots, m. \end{aligned}$$

Then ϕ satisfies (4.1) and (4.2) with respect to ν. Moreover, an easy computation shows that

$$\frac{\| \int_{\mathrm{Cone}(X)} \phi(v)\nu(dv) \|^2}{\int_{\mathrm{Cone}(X)} \|\phi(v)\|^2 \nu(dv)} \leq \frac{\| \int_{\mathrm{Cone}(X)} \phi'(v)\nu'(dv) \|^2}{\int_{\mathrm{Cone}(X)} \|\phi'(v)\|^2 \nu'(dv)}.$$

Hence, by the definition of δ, the latter assertion follows. □

Lemma 4.6. *Let (X, d_X) be a complete* CAT(1) *space and let*

$$\nu = \sum_{i=1}^{m} t_i \mathrm{Dirac}_{[x_i, r_i]} \in \mathcal{P}'(\mathrm{Cone}(X)),$$

where $[x_i, r_i]$ is the point on $\mathrm{Cone}(X)$ represented by $(x_i, r_i) \in X \times [0, \infty)$. Suppose that $\alpha > 0$, $l \in \{1, 2, \cdots, m-1\}$, and

$$\nu' = \frac{1}{\sum_{i=1}^{l} \frac{t_i}{\alpha} + \sum_{i=l+1}^{m} t_i} \left(\sum_{i=1}^{l} \frac{t_i}{\alpha} \mathrm{Dirac}_{[x_i, \alpha r_i]} + \sum_{i=l+1}^{m} t_i \mathrm{Dirac}_{[x_i, r_i]} \right).$$

Then $\mathrm{bar}(\nu') = O_{\mathrm{Cone}(X)}$ if and only if $\mathrm{bar}(\nu) = O_{\mathrm{Cone}(X)}$. Moreover, if $\mathrm{bar}(\nu) = \mathrm{bar}(\nu') = O_{\mathrm{Cone}(X)}$ and if $\alpha > 1$ (resp. $0 < \alpha < 1$), then the inequality $\delta(\nu) \leq \delta(\nu')$ holds if and only if

$$\alpha \frac{\sum_{i=1}^{l} t_i r_i^2}{\sum_{i=l+1}^{m} t_i r_i^2} \leq \frac{\sum_{i=1}^{l} t_i}{\sum_{i=l+1}^{m} t_i} \quad \left(\textit{resp. } \alpha \frac{\sum_{i=1}^{l} t_i r_i^2}{\sum_{i=l+1}^{m} t_i r_i^2} \geq \frac{\sum_{i=1}^{l} t_i}{\sum_{i=l+1}^{m} t_i} \right). \tag{4.3}$$

Proof. The equivalence between $\mathrm{bar}(\nu) = O_{\mathrm{Cone}(X)}$ and $\mathrm{bar}(\nu') = O_{\mathrm{Cone}(X)}$ is an immediate consequence of Lemma 3.2. Assume that $\mathrm{bar}(\nu) = \mathrm{bar}(\nu') = O_{\mathrm{Cone}(X)}$, and fix some real Hilbert space $\mathcal{H}$ of dimension $\geq m$. Then there is a natural bijection $\phi \mapsto \phi'$ between the set of all maps from $\mathrm{supp}(\nu)$ to $\mathcal{H}$ satisfying (4.1) and (4.2) with respect to ν, and the set of all maps from $\mathrm{supp}(\nu')$ to $\mathcal{H}$ satisfying (4.1) and (4.2) with respect to ν': it is given by

$$\phi'[x_i, \alpha r_i] = \alpha\phi[x_i, r_i], \quad i = 1, \cdots, l,$$
$$\phi'[x_i, r_i] = \phi[x_i, r_i], \quad i = l+1, \cdots, m.$$

Let $\phi : \mathrm{supp}(\nu) \to \mathcal{H}$ and $\phi' : \mathrm{supp}(\nu') \to \mathcal{H}$ be the maps satisfying (4.1) and (4.2) with respect to ν and ν' respectively, and corresponding to each other under this bijection. Let

$$T = \frac{1}{\frac{1}{\alpha}\sum_{i=1}^{l} t_i + \sum_{i=l+1}^{m} t_i}.$$

Then we have

$$\frac{\| \int_{\mathrm{Cone}(X)} \phi'(p)\nu'(dp)\|^2}{\int_{\mathrm{Cone}(X)} \|\phi'(p)\|^2\nu'(dp)} = T\frac{\| \sum_{i=1}^{m} t_i\phi[x_i, r_i]\|^2}{\alpha\sum_{i=1}^{l} t_i\|\phi[x_i, r_i]\|^2 + \sum_{i=l+1}^{m} t_i\|\phi[x_i, r_i]\|^2}.$$

Hence,

$$\begin{aligned}
(4.4)\quad & \frac{\| \int_{\mathrm{Cone}(X)} \phi'(p)\nu'(dp)\|^2}{\int_{\mathrm{Cone}(X)} \|\phi'(p)\|^2\nu'(dp)} - \frac{\| \int_{\mathrm{Cone}(X)} \phi(p)\nu(dp)\|^2}{\int_{\mathrm{Cone}(X)} \|\phi(p)\|^2\nu(dp)} \\
&= \left\|\sum_{i=1}^{m} t_i\phi[x_i, r_i]\right\|^2 \frac{T\sum_{i=1}^{m} t_i r_i^2 - \alpha\sum_{i=1}^{l} t_i\|\phi[x_i, r_i]\|^2 - \sum_{i=l+1}^{m} t_i\|\phi[x_i, r_i]\|^2}{\left(\alpha\sum_{i=1}^{l} t_i\|\phi[x_i, r_i]\|^2 + \sum_{i=l+1}^{m} t_i\|\phi[x_i, r_i]\|^2\right)\left(\sum_{i=1}^{m} t_i r_i^2\right)}.
\end{aligned}$$

We also have

$$\begin{aligned}
(4.5)\quad & T\sum_{i=1}^{m} t_i r_i^2 - \alpha\sum_{i=1}^{l} t_i\|\phi[x_i, r_i]\|^2 - \sum_{i=l+1}^{m} t_i\|\phi[x_i, r_i]\|^2 \\
&= \frac{1-\alpha}{(1-\alpha)\left(\sum_{i=1}^{l} t_i\right) + \alpha}\left\{\alpha(\sum_{i=l+1}^{m} t_i)(\sum_{i=1}^{l} t_i r_i^2) - (\sum_{i=1}^{l} t_i)(\sum_{i=l+1}^{m} t_i r_i^2)\right\}.
\end{aligned}$$

By (4.4) and (4.5), the inequality

$$\frac{\| \int_{\mathrm{Cone}(X)} \phi'(p)\nu'(dp)\|^2}{\int_{\mathrm{Cone}(X)} \|\phi'(p)\|^2\nu'(dp)} \geq \frac{\| \int_{\mathrm{Cone}(X)} \phi(p)\nu(dp)\|^2}{\int_{\mathrm{Cone}(X)} \|\phi(p)\|^2\nu(dp)}$$

holds if and only if

$$\alpha \geq 1, \quad \alpha(\sum_{i=l+1}^{m} t_i)(\sum_{i=1}^{l} t_i r_i^2) - (\sum_{i=1}^{l} t_i)(\sum_{i=l+1}^{m} t_i r_i^2) \leq 0$$

or

$$0 < \alpha \leq 1, \quad \alpha(\sum_{i=l+1}^{m} t_i)(\sum_{i=1}^{l} t_i r_i^2) - (\sum_{i=1}^{l} t_i)(\sum_{i=l+1}^{m} t_i r_i^2) \geq 0.$$

The lemma follows easily from this equivalence and the bijectivity of the correspondence $\phi \leftrightarrow \phi'$. □

Proof of Proposition 4.4. First suppose that $\mu \in \mathcal{P}(\mathrm{Cone}(X))$, $\mathrm{bar}(\mu) = O_{\mathrm{Cone}(X)}$, and $\mathrm{supp}(\mu) \subset \iota(X)$. Let $\iota : X \to \mathrm{Cone}(X)$ be the canonical inclusion, and let $\iota^{-1} : \iota(X) \to X$ be the inverse map. Let $\tilde{\phi} : X \to \mathcal{H}$ be a map from X to some Hilbert space $\mathcal{H}$ satisfying (3.2). Then the restriction $\phi = [\tilde{\phi} \circ \iota^{-1}]|_{\mathrm{supp}(\mu)}$ of $\tilde{\phi} \circ \iota^{-1} : \iota(X) \to \mathcal{H}$ to $\mathrm{supp}(\mu)$ satisfies (4.1) and (4.2). Moreover we have

$$\|\mathbb{E}_{\iota_*^{-1}\mu}[\tilde{\phi}]\|^2 = \frac{\| \int_{\mathrm{Cone}(X)} \phi(v)\mu(dv)\|^2}{\int_{\mathrm{Cone}(X)} \|\phi(v)\|^2 \mu(dv)}.$$

Hence by the definitions of $\tilde{\delta}(\iota_*^{-1}\mu)$ and $\delta(\mu)$, we have

$$\delta(\mu) \leq \tilde{\delta}(\iota_*^{-1}\mu).$$

Thus, if we prove the existence of $\nu' \in \mathcal{P}(\mathrm{Cone}(X))$ such that

$$\delta(\nu) \leq \delta(\nu'), \quad \mathrm{supp}(\nu') \subset \iota(X) \tag{4.6}$$

for any

$$\nu = \sum_{i=1}^{m} t_i \mathrm{Dirac}_{[x_i, r_i]} \in \mathcal{P}'(\mathrm{Cone}(X))$$

with $\mathrm{bar}(\nu) = O_{\mathrm{Cone}(X)}$, then the desired assertion follows. Here, we can assume $r_i > 0$ for all $i \in \{1, \cdots, m\}$ by Lemma 4.5. And, if $r_1 = r_2 = \cdots = r_m$, we can take

$$\nu' = \sum_{i=1}^{m} t_i \mathrm{Dirac}_{[x_i, 1]},$$

and ν' satisfies (4.6) because it is straightforward that $\delta(\nu) = \delta(\nu')$. So we can assume $r_1 = \cdots = r_l < r_{l+1} \leq \cdots \leq r_m$ without loss of generality. Then we have

$$\left(\frac{\sum_{i=1}^{l} t_i}{\sum_{i=l+1}^{m} t_i} \right) \Big/ \left(\frac{\sum_{i=1}^{l} t_i r_i^2}{\sum_{i=l+1}^{m} t_i r_i^2} \right) \geq \frac{r_{l+1}^2}{r_1^2} \geq \frac{r_{l+1}}{r_1}.$$

Hence, if we set

$$\nu_0 = \frac{1}{\frac{r_1}{r_{l+1}} \sum_{i=1}^{l} t_i + \sum_{i=l+1}^{m} t_i} \left(\sum_{i=1}^{l} \frac{r_1 t_i}{r_{l+1}} \mathrm{Dirac}_{[x_i, r_{l+1}]} + \sum_{i=l+1}^{m} t_i \mathrm{Dirac}_{[x_i r_i]} \right),$$

then we have

$$\delta(\nu_0) \geq \delta(\nu)$$

by Lemma 4.6. Repeating this procedure, we finally get

$$\nu_1 = \sum_{i=1}^{m} s_i \mathrm{Dirac}_{[x_i, r_m]},$$

which satisfies $\delta(\nu_1) \geq \delta(\nu)$. If we set $\nu' = \sum_{i=1}^{m} s_i \mathrm{Dirac}_{[x_i,1]}$, it is easily seen that $\delta(\nu') = \delta(\nu_1)$, and the assertion follows. $\square$

5. Proof of the theorem

Recall that the Gromov-Hausdorff precompactness is known to be equivalent to the uniformly total boundedness. We call the family $\mathcal{X}$ of metric spaces *uniformly totally bounded* if the following two conditions are satisfied:

- There is a constant D such that $\mathrm{diam}(X) \leq D$ for all $X \in \mathcal{X}$.
- For any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that each $X \in \mathcal{X}$ contains a subset $S_{X,\varepsilon}$ with the following property: the cardinality of $S_{X,\varepsilon}$ is no greater than $N(\varepsilon)$ and X is covered by the union of all ε-balls whose centers are in $S_{X,\varepsilon}$.

By Lemma 4.2, Lemma 4.3 and Proposition 4.4, to prove Theorem 1.1 it suffices to prove the following proposition.

Proposition 5.1. *Let (X, d_X) be a complete $\mathrm{CAT}(1)$ space. Assume that there exist $N \in \mathbb{N}$ and a subset $S = \{x_i\}_{i=1}^{N} \subset X$ such that X is covered by the union of all $\frac{\pi}{12}$-balls whose centers are in S. Then there exists a constant $C(N) < 1$, depending only on N, such that*

$$\tilde{\delta}(X) < C(N).$$

Remark 5.2. It follows from the argument in the proof of Proposition 5.1, we can take

$$C(N) = \left(\frac{2}{3} + \frac{1}{3}\sqrt{\frac{e^{-\frac{\pi^2}{36N}} + 1}{2}} \right)^2 .$$

as a constant $C(N)$ in the proposition.

Before proving Proposition 5.1, we will recall a well-known construction of a map from a Hilbert space to the unit sphere in another Hilbert space, and derive some necessary estimates for them. We follow Dadarlat and Guentner [3] to explain this construction. Let $\mathcal{H}$ be a Hilbert space. Let

$$\mathrm{Exp}(\mathcal{H}) = \mathbb{R} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots,$$

and define $\mathrm{Exp} : \mathcal{H} \to \mathrm{Exp}(\mathcal{H})$ by

$$\mathrm{Exp}(\zeta) = 1 \oplus \zeta \oplus \left(\frac{1}{\sqrt{2!}} \zeta \otimes \zeta \right) \oplus \left(\frac{1}{\sqrt{3!}} \zeta \otimes \zeta \otimes \zeta \right) \oplus \cdots .$$

For $t > 0$, define a map G_t from $\mathcal{H}$ to $\mathrm{Exp}(\mathcal{H})$ to be

$$G_t(\zeta) = e^{-t\|\zeta\|^2} \mathrm{Exp}(\sqrt{2t}\zeta).$$

Then simple computation shows that

$$\cos\angle(G_t(\zeta), G_t(\zeta')) = \langle G_t(\zeta), G_t(\zeta') \rangle = e^{-t\|\zeta - \zeta'\|^2} \tag{5.1}$$

for all $\zeta, \zeta' \in \mathcal{H}$. In particular, $\|G_t(\zeta)\| = 1$ for all $\zeta \in \mathcal{H}$. Hence we can regard G_t as a map from $\mathcal{H}$ to the unit sphere in $\mathrm{Exp}(\mathcal{H})$.

We need the following estimate to prove Proposition 5.1.

Lemma 5.3. *Let (X, d_X) be a metric space, and let $F : X \to \mathcal{H}$ be an L-Lipschitz map ($L > 0$) to some Hilbert space. Suppose that $0 < tL^2 \leq \frac{1}{2}$. Then the map $\phi = G_t \circ F : X \to \mathrm{Exp}(\mathcal{H})$ satisfies*

$$\angle(\phi(x), \phi(y)) \leq \min\{\pi, d_X(x,y)\}$$

for all $x, y \in X$.

Proof. By (5.1) and L-Lipschitz continuity of F, it is sufficient to show that

$$e^{-tL^2 d_X(x,y)^2} \geq \cos\left(\min\{\pi, d_X(x,y)\}\right) \tag{5.2}$$

for all $x, y \in X$ and all $t \in (0, \frac{1}{2L^2})$. When $d_X(x,y) \geq \frac{\pi}{2}$, (5.2) is obvious. So, if we put $a = tL^2$ and $d = d_X(x,y)$, then what we have to show is that

$$a \leq \frac{-\log(\cos d)}{d^2} \tag{5.3}$$

holds for any $a \in (0, \frac{1}{2}]$ and any $d \in [0, \frac{\pi}{2})$. But this is obvious because the right-hand side of (5.3) is non-decreasing with respect to d. $\square$

Now we are ready to prove Proposition 5.1.

Proof of Proposition5.1. First we define a map F_S from X to $\mathbb{R}^N$ by

$$F_S(x) = (d_X(x, x_1), d_X(x, x_2), \cdots d_X(x, x_N))$$

for $x \in X$. Then F_S is $\sqrt{N}$-Lipschitz since

$$\|F_S(x) - F_S(y)\| = \left\{\sum_{i=1}^{N} (d_X(x, x_i) - d_X(y, x_i))^2\right\}^{\frac{1}{2}} \leq \sqrt{N} \cdot d_X(x,y).$$

On the other hand, by the definition of the subset S, for any $x, y \in X$ with $d_X(x,y) \geq \frac{\pi}{3}$, there exist $i_0, i_1 \in \{1, \cdots N\}$ such that

$$d_X(x_{i_0}, x) \geq \frac{\pi}{4}, \quad d_X(x_{i_0}, y) \leq \frac{\pi}{12},$$
$$d_X(x_{i_1}, y) \geq \frac{\pi}{4}, \quad d_X(x_{i_1}, x) \leq \frac{\pi}{12}.$$

Hence

$$\begin{aligned} &\|F_S(x) - F_S(y)\| \\ &\quad \geq \sqrt{(d_X(x_{i_0}, x) - d(x_{i_0}, y))^2 + (d_X(x_{i_1}, x) - d(x_{i_1}, y))^2} \geq \frac{\pi}{3\sqrt{2}} \end{aligned} \tag{5.4}$$

for any $x, y \in X$ with $d_X(x,y) \geq \frac{\pi}{3}$.

We now set $\phi = G_{\frac{1}{2N}} \circ F_S : X \to \mathrm{Exp}(\mathbb{R}^N)$. Then the all values of ϕ are contained in the unit sphere of $\mathrm{Exp}(\mathbb{R}^N)$, and ϕ satisfies

$$\angle(\phi(x), \phi(y)) \leq \min\{\pi, d_X(x,y)\}$$

for all $x, y \in X$ by Lemma 5.3. Moreover (5.1) and (5.4) imply that

$$\angle(\phi(x), \phi(y)) \geq \arccos(e^{-\frac{\pi^2}{36N}}) \tag{5.5}$$

for any $x, y \in X$ with $d_X(x,y) \geq \frac{\pi}{3}$.

Set $\eta = \arccos(e^{-\frac{\pi^2}{36N}})$, and let μ be an arbitrary measure in $\mathcal{P}(X)$ with $\mathrm{bar}(\iota_*\mu) = O_{\mathrm{Cone}(X)}$, where $\iota : X \to \mathrm{Cone}(X)$ is the canonical inclusion and $O_{\mathrm{Cone}(X)}$ is the cone point of $\mathrm{Cone}(X)$. Then we have

$$\phi_*\mu\left(B\left(v, \frac{\eta}{2}\right)\right) \leq \frac{2}{3} \tag{5.6}$$

for any point v on the unit sphere in $\mathrm{Exp}(\mathbb{R}^N)$, where

$$B\left(v, \frac{\eta}{2}\right) = \left\{u \in \mathrm{Exp}(\mathbb{R}^N) \,\middle|\, \|u\| = 1,\ \angle(v,u) < \frac{\eta}{2}\right\}.$$

This is because if there exists some vector $\phi(x_0)$ contained in $B\left(v, \frac{\eta}{2}\right) \cap \phi(X)$, then by (5.5) and Corollary 3.3 we have

$$\begin{aligned}
\phi_*\mu\left(B\left(v, \frac{\eta}{2}\right)\right) &\leq \phi_*\mu\left(B\left(\phi(x_0), \eta\right)\right) \\
&= \mu\left(\phi^{-1}\left(B\left(\phi(x_0), \eta\right)\right)\right) \\
&\leq \mu\left(B\left(x_0, \frac{\pi}{3}\right)\right) \leq \frac{2}{3},
\end{aligned}$$

where $B\left(x_0, \frac{\pi}{3}\right)$ is the open ball in X centered at x_0 with radius $\frac{\pi}{3}$. In the case $B\left(v, \frac{\eta}{2}\right) \cap \phi(X) = \phi$, (5.6) obviously holds.

By (5.6), we have

$$\begin{aligned}
\int_X \langle v, \phi(x)\rangle \mu(dx) &= \int_{\mathcal{S}} \langle v, u\rangle \phi_*\mu(du) \\
&= \int_{B(v,\frac{\eta}{2})} \langle v, u\rangle \phi_*\mu(du) + \int_{\mathcal{S}\setminus B(v,\frac{\eta}{2})} \langle v, u\rangle \phi_*\mu(du) \\
&\leq 1 \times \phi_*\mu\left(B\left(v, \frac{\eta}{2}\right)\right) + \cos\frac{\eta}{2} \times \left\{1 - \phi_*\mu\left(B\left(v, \frac{\eta}{2}\right)\right)\right\} \\
&\leq 1 \times \frac{2}{3} + \left(\cos\frac{\eta}{2}\right) \times \frac{1}{3},
\end{aligned}$$

where $\mathcal{S}$ is the unit sphere in $\mathrm{Exp}(\mathbb{R}^N)$. Setting $v = \tilde{\mathbb{E}}_\mu[\phi]$ in the above inequality and using (3.1), we have

$$\|\mathbb{E}_\mu[\phi]\| = \left\|\int_X \langle \tilde{\mathbb{E}}_\mu[\phi], \phi(x)\rangle \mu(dx)\right\| \leq c_N,$$

where

$$c_N = 1 \times \frac{2}{3} + \left(\cos\frac{\eta}{2}\right) \times \frac{1}{3} = \frac{2}{3} + \frac{1}{3}\sqrt{\frac{e^{-\frac{\pi^2}{36N}} + 1}{2}}$$

Thus, by the definition of $\tilde{\delta}(X)$,

$$\tilde{\delta}(X) \leq c_N^2 < 1$$

which proves the proposition. $\square$

Finally, we remark that the proof of Proposition 5.1 works for the following more general statement.

Proposition 5.4. *Let* $0 < \theta < \frac{\pi}{2}$*,* $0 < \alpha < 1$ *and* $\varepsilon > 0$*. Let* (X, d_X) *be a complete* CAT(1) *space. Assume that there exists a finite subset* $S \subset X$ *such that*

$$\#\left\{s \in S \,\middle|\, \|d_X(x,s) - d_X(y,s)\| \geq \varepsilon\right\} \geq \alpha \# S$$

whenever $x, y \in X$ *and* $d(x,y) \geq \theta$*. Here,* $\#S$ *stands for the cardinality of* S*. Then there exists a constant* $C = C(\theta, \alpha, \varepsilon) < 1$ *such that*

$$\tilde{\delta}(X) \leq C.$$

Proof. We denote the cardinality of S by N. Let F_S be the map from X to $\mathbb{R}^N$ as in the proof of Proposition 5.1 with respect to our set S. Then F_S is $\sqrt{N}$-Lipschitz and we have

$$\|F_S(x) - F_S(y)\| \geq \sqrt{\alpha N}\varepsilon \tag{5.7}$$

for any $x, y \in X$ with $d_X(x,y) \geq \theta$. If we set $\phi = G_{\frac{1}{2N}} \circ F_S : X \to \mathrm{Exp}(\mathbb{R}^N)$, then all the values of ϕ are contained in the unit sphere of $\mathrm{Exp}(\mathbb{R}^N)$, and ϕ satisfies

$$\angle\left(\phi(x), \phi(y)\right) \leq \min\{\pi, d_X(x,y)\}$$

for all $x, y \in X$ by Lemma 5.3. Moreover (5.1) and (5.7) imply that

$$\angle\left(\phi(x), \phi(y)\right) \geq \arccos(e^{-\frac{\alpha\varepsilon^2}{2}}) \tag{5.8}$$

for any $x, y \in X$ with $d_X(x,y) \geq \theta$.

Now the rest of the proof is done exactly in the same manner as in the proof of Proposition 5.1, and we have

$$\tilde{\delta}(X) \leq (c_{\theta,\alpha,\varepsilon})^2,$$

where

$$\begin{aligned} c_{\theta,\alpha,\varepsilon} &= 1 \times \frac{1}{1+\cos\theta} + \left(\cos\frac{\arccos(e^{-\frac{\alpha\varepsilon^2}{2}})}{2}\right) \times \left(1 - \frac{1}{1+\cos\theta}\right) \\ &= \frac{1}{1+\cos\theta} + \sqrt{\frac{e^{-\frac{\alpha\varepsilon^2}{2}}+1}{2}} \times \frac{\cos\theta}{1+\cos\theta} < 1. \end{aligned}$$

□

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